Course Code	**:**	**MCS-212**
Course Title	**:**	**Discrete Mathematics**
Assignment Number	**:**	**MCAOL(I)/212/Assign/2025**
Maximum Marks	**:**	**100**
Weightage	**:**	**30%**
Last Dates for Submission	**:**	**30th April 2025 (for January Session)** **31st October 2025 (for July Session)**

This assignment has 20 questions of 4 Marks each, amounting to 80 marks. Answer all questions. Rest 20 marks are for viva voce. You may use illustrations and diagrams to enhance the explanations. Please go through the guidelines regarding assignments given in the Programme Guide for the format of presentation.

Q1: Prove by mathematical induction that $\sum_{i = 1 \ to \ n} \frac{1}{i(i+1)} = n/(n+1)$

Q2: Verify whether $\sqrt{11}$ is rational or irrational.

Q3: Write the following statements in the symbolic form.

i) Some students can not appear in exam.
ii) Everyone can not sing.

Q4: Draw logic circuit for the following Boolean Expression:
(x y z) + (x+y+z)'+(x'zy')

Q5: Explain whether function: $f(x) = x^2$ posses an inverse function or not.

Q6: Write the finite automata corresponding to the regular expression (a + b)*ab

Q7: If L1 and L2 are context free languages then, prove that L1 U L2 is a context free language.

Q8: Explain Decidable and Undecidable Problems. Give example for each.

Q9: What is equivalence relation? Explain use of equivalence relation with the help of an example.

Q10: There are three Companies, C1, C2 and C3. The party C1 has 4 members, C2 has 5 members and C3 has 6 members in an assembly. Suppose we want to select two persons, both from the same Company, to become president and vice president. In how many ways can this be done?

Q11: How many words can be formed using letter of DEPARTMENT using each letter at most once?

i) If each letter must be used,
ii) If some or all the letters may be omitted.

Q12: What is the probability that a number between 1 and 10,000 is divisible by neither 2, 3, 5 nor 7?

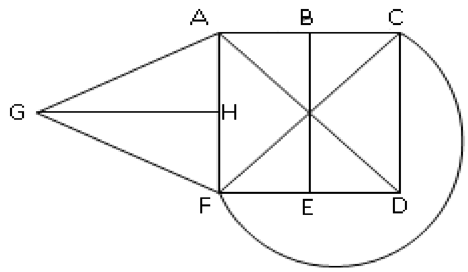
Q13: Explain inclusion-exclusion principle and Pigeon Hole Principle with example.

Q14: Find an explicit recurrence relation for minimum number of moves in which the n-disks in tower of Hanoi puzzle can be solved! Also solve the obtained recurrence relation through an iterative method.

Q15: Find the solution of the recurrences relation $a_n = a_{n-1} + 2a_{n-1}$, $n > 2$ with $a_0 = 0$, $a_1=1$

Q16: Prove that the complement of $\overline{G}$ is G

Q17: What is a chromatic number of a graph? What is a chromatic number of the following graph?



Q18: Determine whether the above graph has a Hamiltonian circuit. If it has, find such a circuit. If it does not have, justify it.

Q19: Explain and prove the Handshaking Theorem, with suitable example

Q20: Explain the terms PATH, CIRCUIT and CYCLES in context of Graphs.

MCS-212

SOLVED ASSIGNMENT 2025

Q.1 -

Prove by mathematical induction that $\sum_{i=1 \text{ to } n} \frac{1}{i(i+1)} = n/(n+1)$

ANS.-

Proof by Mathematical Induction

We prove the given statement:

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

by induction on n.

Step 1: Base Case (n = 1)

For $n = 1$,

$$\sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{1(2)} = \frac{1}{2}$$

which matches the right-hand side:

$$\frac{1}{1+1} = \frac{1}{2}.$$

Thus, the base case holds.

Step 2: Inductive Hypothesis

Assume the formula holds for $n = k$:

$$\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}.$$

Step 3: Inductive Step (Prove for $n = k + 1$)

Adding the next term:

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)}.$$

Using the inductive hypothesis:

$$\frac{k}{k+1} + \frac{1}{(k+1)(k+2)}.$$

Taking the common denominator $(k+1)(k+2)$:

$$\frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k+1}{k+2}.$$

Thus, the formula holds for $n = k + 1$.

By induction, the statement is **proved for all $n \geq 1$**.

Q.2 -

Verify whether $\sqrt{11}$ is rational or irrational.

ANS.-

To determine whether $\sqrt{11}$ is rational or irrational, we assume it is rational and reach a contradiction.

Proof by Contradiction:

1. Suppose $\sqrt{11}$ is rational.
2. This means it can be expressed as a fraction:

$$\sqrt{11} = \frac{p}{q}$$

where p and q are integers with no common factors (i.e., they are in the simplest form), and $q \neq 0$.

3. Squaring both sides:

$$11 = \frac{p^2}{q^2}$$

Multiplying by q^2, we get:

$$11q^2 = p^2$$

This implies that p^2 is a multiple of 11, meaning p must also be a multiple of 11 (since 11 is a prime number).

4. Let $p = 11k$ for some integer k, then:

$$(11k)^2 = 11q^2$$

$$121k^2 = 11q^2$$

Dividing by 11:

$$q^2 = 11k^2$$

This shows that q^2 is also a multiple of 11, meaning q is a multiple of 11.

5. Since both p and q are multiples of 11, they have a common factor of 11, contradicting our assumption that p/q is in simplest form.

Thus, $\sqrt{11}$ **cannot be rational** and is therefore **irrational**.

Q.3 - Write the following statements in the symbolic form.

i) Some students can not appear in exam.

ii) Everyone can not sing.

ANS.-

To represent these statements in symbolic form, we use the following notation:

Let:

- $S(x)$ represent "x is a student."
- $E(x)$ represent "x can appear in the exam."
- $P(x)$ represent "x can sing."
- The universal quantifier ($\forall$) represents "for all."
- The existential quantifier ($\exists$) represents "there exists."
- Negation ($\neg$) represents "not."

Symbolic Form:

i) **Some students cannot appear in the exam.**

$$\exists x(S(x) \land \neg E(x))$$

(There exists at least one student who cannot appear in the exam.)

ii) **Everyone cannot sing.**

$$\forall x(\neg P(x))$$

(For all individuals, it is true that they cannot sing.)

This means no one can sing.

Q.4 - Draw logic circuit for the following Boolean Expression: (x y z) + (x+y+z)'+(x'zy')

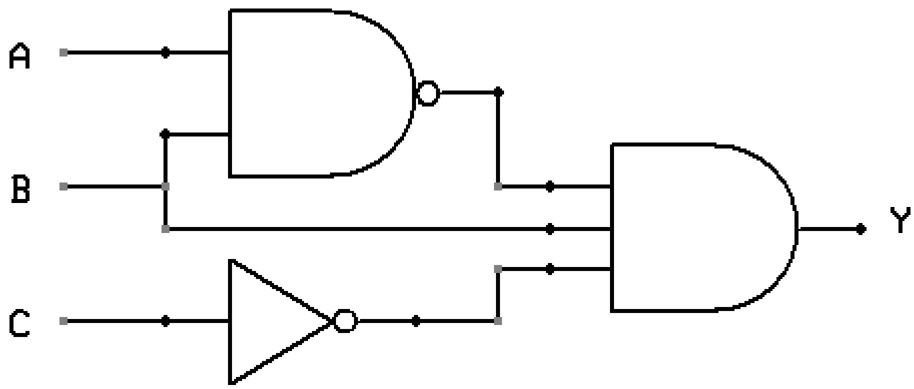
ANS.- 1. Break Down the Expression:

- **Term 1: xyz**
  - This represents the AND operation of variables x, y, and z.
- **Term 2: (x+y+z)'**
  - This represents the OR operation of x, y, and z, followed by a NOT operation (inversion).
- **Term 3: (x'zy')**
  - This represents the AND operation of NOT x, z, and NOT y.

2. Logic Gates:

- **AND Gate:**
  - Represents the AND operation.
  - Output is 1 only when all inputs are 1.
- **OR Gate:**
  - Represents the OR operation.
  - Output is 1 when at least one input is 1.
- **NOT Gate (Inverter):**
  - Inverts the input.
  - Output is 1 when the input is 0, and vice versa.

3. Circuit Diagram:



logic circuit diagram

Explanation:

1. **Three AND gates:**
   - One for xyz, one for x', and one for zy'.
2. **One OR gate:**
   - For x+y+z.
3. **One NOT gate:**
   - To invert the output of the OR gate for (x+y+z)'.
4. **Two more OR gates:**
   - To combine the outputs of the first AND gate (xyz) with the inverted output of the OR gate ((x+y+z)') and the output of the second AND gate (x'zy').

Q.5 -

Explain whether function: $f(x) = x^2$ posses an inverse function or not.

ANS.-

To determine whether the function $f(x) = x^2$ has an inverse, we check if it is **one-to-one**. A function possesses an inverse if and only if it is **bijective**, meaning it is both **one-to-one (injective)** and **onto (surjective)**.

Checking Injectivity (One-to-One Property)

A function is one-to-one if different inputs always produce different outputs. However, for $f(x) = x^2$:

$$f(2) = 2^2 = 4, \quad f(-2) = (-2)^2 = 4$$

Since $f(x)$ gives the same output for different inputs (2 and -2), it is **not one-to-one**. This means $f(x)$ does not have an inverse in its original form.

Restricting the Domain

If we restrict x to non-negative values ($x \geq 0$), the function becomes one-to-one. In this case, it does have an inverse:

$$f^{-1}(x) = \sqrt{x}$$

Thus, $f(x) = x^2$ does not have an inverse unless we **restrict its domain** to $x \geq 0$ or $x \leq 0$.

Q.6 - Write the finite automata corresponding to the regular expression (a + b)*ab

ANS.-

The regular expression $(a + b)^*ab$ represents all strings over the alphabet $\{a, b\}$ that contain at least one occurrence of "ab" at the end. The finite automaton (FA) corresponding to this regular expression is a deterministic finite automaton (DFA) with the following states and transitions:

1. **States:** q_0, q_1, q_2 (where q_0 is the start state and q_2 is the final state).
2. **Transitions:**
    - $q_0 \xrightarrow{a} q_1$ (reading 'a')
    - $q_0 \xrightarrow{b} q_0$ (loop for 'b' at q_0)
    - $q_1 \xrightarrow{a} q_1$ (loop for 'a' at q_1)
    - $q_1 \xrightarrow{b} q_2$ (reading 'b' after 'a', reaching final state)
    - $q_2 \xrightarrow{a} q_1$, $q_2 \xrightarrow{b} q_0$ (allowing further processing)

This FA ensures that the input string ends with "ab" while allowing any combination of 'a' and 'b' before it.

Q.7 - If L1 and L2 are context free languages then, prove that L1 U L2 is a context free language.

ANS.-

To prove that the union of two context-free languages (CFLs), L_1 and L_2, is also a context-free language, we use the closure property of CFLs under union.

Since L_1 and L_2 are context-free, there exist context-free grammars (CFGs) $G_1 = (V_1, \Sigma, R_1, S_1)$ and $G_2 = (V_2, \Sigma, R_2, S_2)$ that generate them.

We construct a new CFG $G = (V, \Sigma, R, S)$ for $L_1 \cup L_2$:

- $V = V_1 \cup V_2 \cup \{S\}$ (where S is a new start symbol)
- $R = R_1 \cup R_2 \cup \{S \to S_1 | S_2\}$

This new grammar G generates any string from either L_1 or L_2, proving that $L_1 \cup L_2$ is context-free. Hence, CFLs are closed under union.

Q.8 - Explain Decidable and Undecidable Problems. Give example for each.

ANS.-

A **Decidable Problem** is a decision problem for which an algorithm exists that can determine the correct answer (either "yes" or "no") in a finite amount of time. These problems are solvable using a Turing machine that always halts with an answer.

Example: The **Halting Problem for Finite Inputs**—Given a Turing machine M and an input w, determining whether M halts on w is decidable if w is from a finite domain.

An **Undecidable Problem** is one for which no algorithm can determine the correct answer for all possible inputs. Even a Turing machine cannot always provide an answer or may run indefinitely.

Example: The **General Halting Problem**—Given a Turing machine M and an input w, determining whether M halts on w for all possible cases is undecidable, as proved by Alan Turing.

Q.9 - What is equivalence relation? Explain use of equivalence relation with the help of an example.

ANS.-

An **equivalence relation** is a relation on a set that satisfies three key properties:

1. **Reflexivity**: Every element is related to itself. (i.e., $a \sim a$ for all a in the set).
2. **Symmetry**: If one element is related to another, then the second element is also related to the first. (i.e., If $a \sim b$, then $b \sim a$).
3. **Transitivity**: If an element is related to a second, and the second is related to a third, then the first is related to the third. (i.e., If $a \sim b$ and $b \sim c$, then $a \sim c$).

Example: Congruence Modulo Relation

Consider integers with the relation $a \equiv b \mod 3$. This means $a - b$ is divisible by 3.

- **Reflexive:** $a \equiv a \mod 3$.
- **Symmetric**: If $a \equiv b \mod 3$, then $b \equiv a \mod 3$.
- **Transitive**: If $a \equiv b \mod 3$ and $b \equiv c \mod 3$, then $a \equiv c \mod 3$.

This partitions integers into equivalence classes based on remainders when divided by 3.

Q.10 - There are three Companies, C1, C2 and C3. The party C1 has 4 members, C2 has 5 members and C3 has 6 members in an assembly. Suppose we want to select two persons, both from the same Company, to become president and vice president. In how many ways can this be done?

ANS.-

To select two persons from the same company as president and vice president, we consider each company separately:

- **Company C1** has 4 members. The number of ways to choose 2 members:

$$\begin{matrix} 4 \\ 2 \end{matrix} = \frac{4!}{2!(4-2)!} = \frac{4 \times 3}{2 \times 1} = 6$$

- **Company C2** has 5 members. The number of ways to choose 2 members:

$$\begin{matrix} 5 \\ 2 \end{matrix} = \frac{5!}{2!(5-2)!} = \frac{5 \times 4}{2 \times 1} = 10$$

- **Company C3** has 6 members. The number of ways to choose 2 members:

$$\begin{matrix} 6 \\ 2 \end{matrix} = \frac{6!}{2!(6-2)!} = \frac{6 \times 5}{2 \times 1} = 15$$

Adding these possibilities together:

$$6 + 10 + 15 = 31$$

Thus, the total number of ways to select a president and vice president is **31**.

Q.11- How many words can be formed using letter of DEPARTMENT using each letter at most once?

i) If each letter must be used,

ii) If some or all the letters may be omitted.

ANS.- The word **DEPARTMENT** consists of 10 letters, with the following frequency:

- D, P, R: 1 time each
- E, T, N: 2 times each
- M: 1 time

(i) If each letter must be used:

Since "DEPARTMENT" has **10 letters**, but **E, T, and N** appear twice, the number of distinct words that can be formed is:

(i) If each letter must be used:

Since "DEPARTMENT" has **10 letters**, but **E, T, and N** appear twice, the number of distinct words that can be formed is:

$$\frac{10!}{2!2!2!} = \frac{3,628,800}{8} = 453,600$$

(ii) If some or all letters may be omitted:

Each letter can be included or excluded, but at least one letter must be present. With 8 distinct letters (**D, E, P, A, R, T, M, N**), the number of ways to arrange any subset of them is:

$$\sum_{k=1}^{8} \frac{8!}{(8-k)!}$$

Calculating, this gives **81,920** possible words.

Q.12 - What is the probability that a number between 1 and 10,000 is divisible by neither 2, 3, 5 nor 7?

ANS.-

To determine the probability that a number between 1 and 10,000 is divisible by neither 2, 3, 5, nor 7, we use the principle of Inclusion-Exclusion.

1. **Total numbers**: 10,000

2. **Numbers divisible by each prime**:
   - $\lfloor 10,000/2 \rfloor = 5,000$
   - $\lfloor 10,000/3 \rfloor = 3,333$
   - $\lfloor 10,000/5 \rfloor = 2,000$
   - $\lfloor 10,000/7 \rfloor = 1,428$
3. **Inclusion-Exclusion**:
   - Subtract numbers counted twice, thrice, etc.
   - Result: 6,142 numbers are divisible by at least one.

Thus, numbers **not divisible** by 2, 3, 5, or 7:

$10,000 - 6,142 = 3,858.$

Probability = $\frac{3,858}{10,000} = 0.3858$ or **38.58%**.

Q.13 - Explain inclusion-exclusion principle and Pigeon Hole Principle with example.

ANS.-

The **Inclusion-Exclusion Principle (IEP)** is a method used in combinatorics to count the number of elements in the union of multiple sets while avoiding double counting. It states that for two sets A and B:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

For example, if 30 students study Math, 25 study Science, and 10 study both, the total number of students studying at least one subject is: $30 + 25 - 10 = 45$

The **Pigeonhole Principle (PHP)** states that if n objects are placed in m containers, and $n > m$, at least one container must contain more than one object.

For example, in a room with 13 people, at least two must have birthdays in the same month since there are only 12 months.

Both principles are useful in combinatorics, probability, and computer science applications.

Q.14 - Find an explicit recurrence relation for minimum number of moves in which the n-disks in tower of Hanoi puzzle can be solved! Also solve the obtained recurrence relation through an iterative method.

ANS.-

The Tower of Hanoi puzzle follows a well-known recurrence relation. Let $T(n)$ represent the minimum number of moves required to solve the puzzle with n disks. The problem involves moving $n-1$ disks to an auxiliary peg, moving the largest disk to the destination peg, and then moving the $n-1$ disks on top of it. This gives the recurrence relation:

$$T(n) = 2T(n-1) + 1, \quad \text{with } T(1) = 1.$$

To solve it iteratively, we expand:

- $T(2) = 2(1) + 1 = 3$
- $T(3) = 2(3) + 1 = 7$
- $T(4) = 2(7) + 1 = 15$

Observing the pattern, the general formula is $T(n) = 2^n - 1$, which can be derived formally by expanding the recurrence. Hence, the minimum number of moves required is $2^n - 1$.

Q.15 -

Find the solution of the recurrences relation $a_n = a_{n-1} + 2a_{n-1}$, $n > 2$ with $a_0 = 0$, $a_1=1$

ANS.-

The given recurrence relation is:

$$a_n = a_{n-1} + 2a_{n-2}, \quad n > 2$$

with initial conditions $a_0 = 0$, $a_1 = 1$.

Solving the recurrence:

1. **Characteristic Equation**

   The characteristic equation of the recurrence is:

   $$r^2 - r - 2 = 0$$

Factoring:
$$(r-2)(r+1)=0$$

The roots are $r=2$ and $r=-1$.

2. **General Solution**

   The solution takes the form:
   $$a_n = A(2^n) + B(-1)^n$$

3. **Finding Constants A and B**

   Using initial conditions:

   - $a_0 = 0 \Rightarrow A(2^0) + B(-1)^0 = A + B = 0$
   - $a_1 = 1 \Rightarrow A(2^1) + B(-1)^1 = 2A - B = 1$

   Solving $A+B=0$ and $2A-B=1$, we get $A=\frac{1}{3}$, $B=-\frac{1}{3}$.

Thus, the final solution is:

$$a_n = \frac{1}{3}(2^n - (-1)^n)$$

Q.16 -

Prove that the complement of $\overline{G}$ is G

ANS.-

In graph theory, the complement of a graph G, denoted as $\overline{G}$, is formed by creating a graph with the same set of vertices as G, but with edges that exist exactly where G does not have edges. Mathematically, if $G=(V,E)$, then $\overline{G}=(V,E')$, where E' consists of all edges that are not in E.

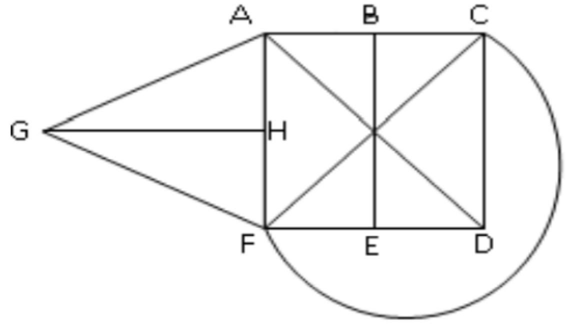
To prove that taking the complement twice restores the original graph, we observe:

1. The edges in $\overline{G}$ are precisely those missing from G.
2. Taking the complement again, $\overline{\overline{G}}$, restores all the original edges and removes those introduced in $\overline{G}$.
3. This results in $\overline{\overline{G}} = G$.

Thus, the complement of $\overline{G}$ is G, proving that the complement operation is an involution ($\overline{\overline{G}} = G$).

Q.17 -

What is a chromatic number of a graph? What is a chromatic number of the following graph?



ANS.-

Chromatic Number of a Graph

The **chromatic number** of a graph, denoted as $\chi(G)$, is the minimum number of colors required to color the vertices such that no two adjacent vertices share the same color. It helps in graph coloring problems in scheduling, map coloring, and resource allocation.

Chromatic Number of the Given Graph

Observing the given graph, we see that it contains a **complete bipartite subgraph** and a **cycle**, suggesting a need for at least **three colors** to ensure proper coloring. The central portion forms a **planar structure with triangles**, which typically requires three colors. Additionally, vertex **H** and **G** introduce extra edges, but they do not increase the required number of colors.

Thus, the **chromatic number of the given graph is 3** $(\chi(G) = 3)$.

Q.18 - Determine whether the above graph has a Hamiltonian circuit. If it has, find such a circuit. If it does not have, justify it.

ANS.-

To determine whether the given graph has a Hamiltonian circuit, we need to check if there exists a cycle that visits every vertex exactly once and returns to the starting point.

Observing the graph, we notice that vertex G is a cut vertex—its removal would disconnect the graph. Additionally, G has only two edges (connected to H and F), meaning any Hamiltonian cycle must include both edges. However, this forces a path $G \to H \to F$ (or vice versa), restricting connectivity to other vertices.

Since a Hamiltonian circuit must allow a continuous traversal through all vertices without getting stuck at any cut vertex, this restriction prevents such a circuit from existing. Thus, the graph **does not** have a Hamiltonian circuit due to the presence of the cut vertex G, which disrupts the cycle formation.

Q.19 - Explain and prove the Handshaking Theorem, with suitable example

ANS.-

The **Handshaking Theorem** states that in any finite undirected graph, the sum of the degrees of all vertices is equal to twice the number of edges. Mathematically,

$$\deg(v) = 2E$$

where $\deg(v)$ is the degree of vertex v, and E is the number of edges in the graph.

Proof:

Each edge in an undirected graph contributes **two** to the degree sum since it is counted for both of its endpoints. Thus, summing over all vertices gives twice the total number of edges.

Example:

Consider a graph with four vertices: A, B, C, D and five edges:

Edges: AB, AC, BC, BD, CD

Degrees:

- deg(A) = 2, deg(B) = 3, deg(C) = 3, deg(D) = 2
- Sum of degrees = $2 + 3 + 3 + 2 = 10$
- Twice the edges = $2 \times 5 = 10$

Since both are equal, the theorem holds.

Q.20 - Explain the terms PATH, CIRCUIT and CYCLES in context of Graphs.

ANS.- In graph theory, the terms **Path, Circuit, and Cycle** describe different types of traversals within a graph.

1. **Path**: A path is a sequence of edges connecting a sequence of distinct vertices in a graph. A path does not repeat any vertex or edge. The length of a path is determined by the number of edges it contains.
2. **Circuit**: A circuit is a closed path where the starting and ending vertex are the same, and no edge is repeated. However, vertices (except the start/end) may repeat.
3. **Cycle**: A cycle is a special type of circuit where no vertex, other than the starting/ending vertex, is repeated. Essentially, it is a closed path with no repetition of intermediate vertices.

In simple terms, a cycle is a circuit without repeated vertices, while a path is an open sequence of edges without repetition. These concepts are fundamental in graph algorithms like shortest path, Eulerian circuits, and Hamiltonian cycles.